

Discrete Convexity in Joint Winner Property

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Abstract

In this paper, we reveal a relation between joint winner property (JWP) in the field of valued constraint satisfaction problems (VCSPs) and M^\natural -convexity in the field of discrete convex analysis (DCA). We introduce the M^\natural -convex completion problem, and show that a function f satisfying JWP is Z-free if and only if a certain function \bar{f} associated with f is M^\natural -convex completable. This means that if a function is Z-free, then the function can be minimized in polynomial time via M^\natural -convex intersection algorithms. Furthermore we propose a new algorithm for Z-free function minimization, which is faster than previous algorithms for some parameter values.

1 Introduction

A *valued constraint satisfaction problem (VCSP)* is a general and important framework for discrete optimization (see [14] for details). Informally, the VCSP framework deals with the minimization problem of a function represented as the sum of “small” arity functions. An important line of VCSP research is to investigate what instances are NP-hard and what instances are solvable in polynomial time. Cooper–Žitný [1] showed that if a function represented as the sum of unary or binary functions satisfies the *joint winner property (JWP)*, then the function can be minimized in polynomial time.

Discrete convex analysis (DCA) [7] is a theory of convex functions on the integer lattice. It is well known that a variety of polynomially solvable problems in discrete optimization can be regarded as being related to discrete convexity. *M^\natural -convex functions* play an important role in DCA, appearing in many areas such as operations research, economics, and game theory (see e.g., [7, 8, 9]).

The results of this paper are summarized as follows:

- We reveal a relation between JWP and M^\natural -convexity. That is, we give a DCA interpretation of polynomial-time solvability of JWP.
- To describe the connection of JWP and M^\natural -convexity, we introduce the *M^\natural -convex completion problem*, and give a characterization of M^\natural -convex completability.
- By utilizing a DCA interpretation of JWP, we propose a new algorithm for Z-free function minimization, which is faster than previous algorithms for some parameter values.

This study will hopefully be the first step towards fruitful interactions between VCSPs and DCA.

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Notations. Let \mathbf{R} and \mathbf{R}_+ denote the sets of reals and nonnegative reals, respectively. In this paper, functions can take the infinite value $+\infty$, where $a < +\infty$, $a + \infty = +\infty$ for $a \in \mathbf{R}$, and $0 \cdot (+\infty) = 0$. Let $\overline{\mathbf{R}} := \mathbf{R} \cup \{+\infty\}$ and $\overline{\mathbf{R}}_+ := \mathbf{R}_+ \cup \{+\infty\}$. For a function $f : \{0, 1\}^n \rightarrow \overline{\mathbf{R}}$, the effective domain is denoted as $\text{dom } f := \{x \in \{0, 1\}^n \mid f(x) < +\infty\}$. For a positive integer k , we define $[k] := \{1, 2, \dots, k\}$. For $x = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n$, we define $\text{supp}^+(x) := \{i \in [n] \mid x_i > 0\}$.

2 Preliminaries

Joint Winner Property. Let $d_i \geq 2$ be a positive integer and $D_i := [d_i]$ for $i \in [r]$. We consider a function $f : D_1 \times D_2 \times \dots \times D_r \rightarrow \overline{\mathbf{R}}_+$ represented as the sum of unary or binary functions as

$$f(x_1, x_2, \dots, x_r) = \sum_{i \in [r]} c_i(x_i) + \sum_{1 \leq i < j \leq r} c_{ij}(x_i, x_j), \quad (1)$$

where $c_i : D_i \rightarrow \overline{\mathbf{R}}_+$ is a unary function for $i \in [r]$ and $c_{ij} : D_i \times D_j \rightarrow \overline{\mathbf{R}}_+$ is a binary function for $1 \leq i < j \leq r$. Furthermore we assume $c_{ij} = c_{ji}$ for distinct $i, j \in [r]$. A function f of the form (1) is said [1] to satisfy the *joint winner property (JWP)* if it holds that

$$c_{ij}(a, b) \geq \min\{c_{jk}(b, c), c_{ik}(a, c)\} \quad (2)$$

for all distinct $i, j, k \in [r]$ and all $a \in D_i, b \in D_j, c \in D_k$. A function f of the form (1) satisfying JWP is said to be *Z-free* if it satisfies that

$$|\arg \min\{c_{ij}(a, c), c_{ij}(a, d), c_{ij}(b, c), c_{ij}(b, d)\}| \geq 2 \quad (3)$$

for any $i, j \in [r]$ ($i \neq j$), $\{a, b\} \subseteq D_i$ ($a \neq b$), and $\{c, d\} \subseteq D_j$ ($c \neq d$).

Cooper–Živný [1] showed that if f of the form (1) satisfies JWP, then f can be minimized in polynomial time. In fact, they showed that if f satisfies JWP, then f can be transformed into a certain Z-free function f' in polynomial time such that a minimizer of f' is also a minimizer of f . Moreover they showed that a Z-free function can be minimized in polynomial time.

M^\natural -Convexity. A function $f : \{0, 1\}^n \rightarrow \overline{\mathbf{R}}$ is said [7, 8] to be *M^\natural -convex* if for all $x, y \in \{0, 1\}^n$ and all $i \in \text{supp}^+(x - y)$ there exists $j \in \text{supp}^+(y - x) \cup \{0\}$ such that

$$f(x) + f(y) \geq f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j), \quad (4)$$

where χ_i is the i th unit vector and χ_0 is the zero vector. A function $f : \{0, 1\}^n \rightarrow \overline{\mathbf{R}}$ is said [7] to be *M_2^\natural -convex* if f can be represented as the sum of two M^\natural -convex functions. It is well known that M^\natural -convex functions can be minimized in polynomial time. Furthermore if we are given two M^\natural -convex functions g and h , we can minimize an M_2^\natural -convex function $f = g + h$ in polynomial time by solving the so-called “ M^\natural -convex intersection problem.”

Theorem 1 ([11, Theorem 10][13, Theorem 6.5]; see also [9, Theorem 3.3]). *A function $f : \{0, 1\}^n \rightarrow \overline{\mathbf{R}}$ with the zero vector in $\text{dom } f$ is M^\natural -convex if and only if f satisfies the following two conditions:*

Condition 1: *For all distinct $i, j, k \in [n]$ and all $z \in \{0, 1\}^n$ with $\text{supp}^+(z) \subseteq [n] \setminus \{i, j, k\}$, it holds that*

$$f(z + \chi_i + \chi_j) + f(z + \chi_k) \geq \min\{f(z + \chi_j + \chi_k) + f(z + \chi_i), f(z + \chi_i + \chi_k) + f(z + \chi_j)\}. \quad (5)$$

Condition 2: For all distinct $i, j \in [n]$ and all $z \in \{0, 1\}^n$ with $\text{supp}^+(z) \subseteq [n] \setminus \{i, j\}$, it holds that

$$f(z + \chi_i + \chi_j) + f(z) \geq f(z + \chi_i) + f(z + \chi_j).$$

We pay special attention to quadratic M^\natural -convex functions. Using Theorem 1, we provide a necessary and sufficient condition for the M^\natural -convexity of a function $f : \{0, 1\}^n \rightarrow \overline{\mathbf{R}}$ of the form

$$f(x_1, x_2, \dots, x_n) := \sum_{i \in [n]} h_i x_i + \sum_{1 \leq i < j \leq n} h_{ij} x_i x_j \quad ((x_1, x_2, \dots, x_n) \in \{0, 1\}^n), \quad (6)$$

where we assume $h_{ij} = h_{ji}$ and $h_i < +\infty$ for $i, j \in [n]$.

Lemma 2. A function f of the form (6) is M^\natural -convex if and only if it satisfies the following:

- $h_{ij} \geq \min\{h_{ik}, h_{jk}\} \quad (i, j, k : \text{distinct}).$
- $h_{ij} \geq 0 \quad (i, j : \text{distinct}).$

In Lemma 2, h_{ij} can take the infinite value $+\infty$, whereas all h_{ij} 's are assumed to be finite in the characterization in [5] and [8]. In particular, we refer to the first condition $h_{ij} \geq \min\{h_{ik}, h_{jk}\}$ ($i, j, k : \text{distinct}$) as the *anti-ultrametric property*. Note that no conditions are imposed on h_i . The proof of Lemma 2 is in Section 5

By Lemma 2, we know that M^\natural -convexity of a function of the form (6) depends only on quadratic coefficients $(h_{ij})_{i,j \in [n]}$. We say that a function f of the form (6) is defined by $(h_{ij})_{i,j \in [n]}$ if the quadratic coefficients of f is equal to $(h_{ij})_{i,j \in [n]}$.

3 M^\natural -Convexity in Joint Winner Property

M^\natural -Convex Completion Problem. We introduce the *M^\natural -convex completion problem*, and give a characterization of an M^\natural -convex completable function on $\{0, 1\}^n$ defined by $(h_{ij})_{i,j \in [n]}$. The M^\natural -convex completion problem is the following:

Given: $(h_{ij})_{i,j \in [n]}$ such that $h_{ij} \in \overline{\mathbf{R}}$ or h_{ij} is undefined for every distinct $i, j \in [n]$.

Question: By assigning appropriate values in $\overline{\mathbf{R}}$ to “undefined” elements of $(h_{ij})_{i,j \in [n]}$, can we construct an M^\natural -convex function $f : \{0, 1\}^n \rightarrow \overline{\mathbf{R}}$ of the form (6)?

It should be clear that a defined element can be equal to $+\infty$ and the infinite value $(+\infty)$ may be assigned to undefined elements. If there is an appropriate assignment of $(h_{ij})_{i,j \in [n]}$, then $(h_{ij})_{i,j \in [n]}$ is said to be *M^\natural -convex completable*. If $h_{ij} < 0$ or $h_{ij} < \min\{h_{jk}, h_{ik}\}$ holds for some defined elements h_{ij}, h_{jk}, h_{ik} , then we obviously know that $(h_{ij})_{i,j \in [n]}$ is not M^\natural -convex completable. Hence in considering the M^\natural -convex completion problem, we assume that

$$h_{ij} \geq 0, \quad (7)$$

$$h_{ij} \geq \min\{h_{jk}, h_{ik}\} \quad (8)$$

for all defined elements h_{ij}, h_{jk}, h_{ik} .

For quadratic coefficients $H := (h_{ij})_{i,j \in [n]}$ containing undefined elements, we define the *assignment graph* of H as a graph $G_H = ([n], E_H; w)$, where $E_H := \{\{i, j\} \mid i \neq j \text{ and } h_{ij} \text{ is defined}\}$ and $w : E_H \rightarrow \overline{\mathbf{R}}_+$ is defined by $w(\{i, j\}) := h_{ij}$ for $\{i, j\} \in E_H$. Then the following theorem holds.

Theorem 3. $H := (h_{ij})_{i,j \in [n]}$ is M^\natural -convex completable if and only if $|\arg \min_{e \in C} w(e)| \geq 2$ holds for every chordless cycle C of G_H .

The proof of Theorem 3 is in Section 5.

Remark 4. Farach–Kannan–Warnow [3] introduced the *matrix sandwich problem for ultrametric property*, which contains the M^\natural -convex completion problem as a special case. They also constructed an $O(m + n \log n)$ -time algorithm for the matrix sandwich problem for ultrametric property, where m is the number of defined elements. By using this algorithm, we can obtain an appropriate M^\natural -convex completion in $O(m + n \log n)$ time if one exists.

In this paper, we present a graphic characterization of M^\natural -convex compleatability. With this characterization, we provide a DCA interpretation of polynomial-time solvability of JWP.

Transformation into a Function over $\{0, 1\}$. To connect JWP and M^\natural -convexity, we introduce a transformation of a function $f : D_1 \times D_2 \times \cdots \times D_r \rightarrow \overline{\mathbf{R}}$ into a function $\hat{f} : \{0, 1\}^U \rightarrow \overline{\mathbf{R}}$, where U is the set of all assignments to variables, that is,

$$U := \{(1, 1), (1, 2), \dots, (1, d_1), (2, 1), (2, 2), \dots, (2, d_2), \dots, (r, 1), (r, 2), \dots, (r, d_r)\}.$$

We consider the following correspondence between $x = (x_1, x_2, \dots, x_r) \in D_1 \times D_2 \times \cdots \times D_r$ and $\hat{x} = (\hat{x}_{(1,1)}, \dots, \hat{x}_{(1,d_1)}, \hat{x}_{(2,1)}, \dots, \hat{x}_{(2,d_2)}, \dots, \hat{x}_{(r,1)}, \dots, \hat{x}_{(r,d_r)}) \in \{0, 1\}^U$:

$$(x_1, x_2, \dots, x_r) \mapsto (\underbrace{0, \dots, 0, \overset{(1,x_1)}{\hat{1}}, 0, \dots, 0}_{d_1}, \underbrace{0, \dots, 0, \overset{(2,x_2)}{\hat{1}}, 0, \dots, 0}_{d_2}, \dots, \underbrace{0, \dots, 0, \overset{(r,x_r)}{\hat{1}}, 0, \dots, 0}_{d_r}). \quad (9)$$

That is, $\hat{x}_{(i,a)} = 1$ means that we assign a to x_i , and $\hat{x}_{(i,a)} = 0$ means that we do not. In view of (9), define a function \hat{f} by

$$\hat{f}(\hat{x}) := \begin{cases} f(x) & \text{if there exists } x \text{ satisfying (9),} \\ +\infty & \text{otherwise} \end{cases} \quad (\hat{x} \in \{0, 1\}^U).$$

Note that minimizing f is equivalent to minimizing \hat{f} .

Now we consider the transformation of f of the form (1) into \hat{f} , where f is given in terms of c_i for $i \in [r]$ and c_{ij} for $i, j \in [r]$. We define $\bar{f} : \{0, 1\}^U \rightarrow \overline{\mathbf{R}}$ by

$$\bar{f}(\hat{x}) := \sum_{(i,a) \in U} c_i(a) \hat{x}_{(i,a)} + \sum_{(i,a),(j,b) \in U, (i,a) \neq (j,b)} h_{(i,a),(j,b)} \hat{x}_{(i,a)} \hat{x}_{(j,b)} \quad (\hat{x} \in \{0, 1\}^U), \quad (10)$$

where

$$h_{(i,a),(j,b)} := \begin{cases} c_{ij}(a, b) & \text{if } i \neq j, \\ \text{undefined} & \text{if } i = j. \end{cases} \quad (11)$$

We also define $\delta_U : \{0, 1\}^U \rightarrow \overline{\mathbf{R}}$ by

$$\delta_U(\hat{x}) := \begin{cases} 0 & \text{if there exists } x \text{ satisfying (9),} \\ +\infty & \text{otherwise} \end{cases} \quad (\hat{x} \in \{0, 1\}^U),$$

which is the indicator function for the feasible assignments. Then we have

$$\hat{f}(\hat{x}) = \bar{f}(\hat{x}) + \delta_U(\hat{x}) \quad (\hat{x} \in \{0, 1\}^U).$$

It is clear that δ_U is M^\natural -convex ($\text{dom } \delta_U$ is the base family of a partition matroid, which is a direct sum of matroids of rank 1). Moreover, the values assigned to the undefined elements in (11) do not affect the value of \hat{f} . Hence if $(h_{(i,a),(j,b)})_{(i,a),(j,b) \in U}$ has an M^\natural -convex completion $(\tilde{h}_{(i,a),(j,b)})_{(i,a),(j,b) \in U}$, then \bar{f} defined by $(\tilde{h}_{(i,a),(j,b)})_{(i,a),(j,b) \in U}$ is M^\natural -convex and $\hat{f} = \bar{f} + \delta_U$ is M_2^\natural -convex. This means that \hat{f} can be minimized in polynomial time. We need the values of $(\tilde{h}_{(i,a),(j,b)})_{(i,a),(j,b) \in U}$ in a minimization algorithm of M_2^\natural -convex functions.

A function of the form (1) satisfies JWP if and only if it holds that $h_{(i,a),(j,b)} \geq 0$ and $h_{(i,a),(j,b)} \geq \min\{h_{(j,b),(k,c)}, h_{(i,a),(k,c)}\}$ for defined elements $h_{(i,a),(j,b)}, h_{(j,b),(k,c)}, h_{(i,a),(k,c)}$ given in (10). This means that $(h_{(i,a),(j,b)})_{(i,a),(j,b) \in U}$ satisfies the assumptions (7) and (8) for the M^\natural -convex completion problem. Theorem 3 implies the following theorem (the proof is in Section 5).

Theorem 5. *For a function f of the form (1), let $(h_{(i,a),(j,b)})_{(i,a),(j,b) \in U}$ be defined by (11). Then $(h_{(i,a),(j,b)})_{(i,a),(j,b) \in U}$ is M^\natural -convex completable if and only if f is Z-free.*

4 Algorithm

By using a general algorithm for the M^\natural -convex intersection (minimization of M_2^\natural -convex functions), we can minimize Z-free functions of the form (1) in polynomial time. Suppose that we are given $c_i : D_i \rightarrow \mathbf{R}_+$ for $i \in [r]$, $c_{ij} : D_i \times D_j \rightarrow \overline{\mathbf{R}}_+$ for $1 \leq i < j \leq r$, and a Z-free function f defined as (1). We can minimize f by minimizing $\hat{f} = \bar{f} + \delta_U$ with an M^\natural -convex intersection algorithm.

Here we take advantage of the fact that all the vectors in $\text{dom } \delta_U$ have a constant component sum, i.e., $\sum_{(i,a) \in U} \hat{x}_{(i,a)} = r$ for all $\hat{x} \in \text{dom } \delta_U$. This implies that δ_U is an M-convex function [7] and we can use an *M-convex intersection* algorithm, which is slightly simpler to describe. Since the functions are defined on $\{0, 1\}^n$, the proposed algorithm is actually a variant of valuated matroid intersection algorithms [6]. Specifically, let \bar{f}' denote the restriction of \bar{f} to the hyperplane containing $\text{dom } \delta_U$, i.e.,

$$\bar{f}'(\hat{x}) := \begin{cases} \bar{f}(\hat{x}) & \text{if } \sum_{(i,a) \in U} \hat{x}_{(i,a)} = r, \\ +\infty & \text{otherwise.} \end{cases}$$

Then minimizing $\bar{f} + \delta_U$ is equivalent to minimizing $\bar{f}' + \delta_U$, where \bar{f}' and δ_U are M-convex functions.

The proposed algorithm consists of three steps.

Step 1: On the basis of Theorem 5, we construct an M^\natural -convex function $\bar{f} : \{0, 1\}^U \rightarrow \overline{\mathbf{R}}$ in (10) through an M^\natural -convex completion of $(h_{(i,a),(j,b)})_{(i,a),(j,b) \in U}$ in (11).

Step 2: We find a minimizer of \bar{f}' , to be used as an initial solution in Step 3.

Step 3: We find a minimizer of $\bar{f}' + \delta_U$ by the successive shortest path algorithm with potentials for the M-convex intersection [7] (see also [6, Section 5.2]).

In Step 3 of the algorithm, we use the *auxiliary graph* $G_{\hat{x}, \hat{y}} = (V, E_{\hat{x}, \hat{y}})$ defined for $\hat{x} \in \text{dom } \bar{f}'$

and $\hat{y} \in \text{dom } \delta_U$ by

$$V := \{s, t\} \cup U, \quad (12)$$

$$E_{\hat{x}} := \{((i, a), (j, b)) \mid (i, a), (j, b) \in U, \hat{x} + \chi_{(j, b)} - \chi_{(i, a)} \in \text{dom } \bar{f}'\}, \quad (13)$$

$$E_{\hat{y}} := \{((i, a), (j, b)) \mid (i, a), (j, b) \in U, \hat{y} + \chi_{(i, a)} - \chi_{(j, b)} \in \text{dom } \delta_U\}, \quad (14)$$

$$E^+ := \{(s, (i, a)) \mid (i, a) \in \text{supp}^+(\hat{x} - \hat{y})\}, \quad (15)$$

$$E^- := \{((j, b), t) \mid (j, b) \in \text{supp}^+(\hat{y} - \hat{x})\}, \quad (16)$$

$$E_{\hat{x}, \hat{y}} := E_{\hat{x}} \cup E_{\hat{y}} \cup E^+ \cup E^- \quad (17)$$

with the arc length function $\ell = \ell_{\hat{x}, \hat{y}} : E_{\hat{x}, \hat{y}} \rightarrow \mathbf{R}$ given by

$$\ell(u, v) := \begin{cases} \bar{f}'(\hat{x} + \chi_v - \chi_u) - \bar{f}'(\hat{x}) & \text{if } (u, v) \in E_{\hat{x}}, \\ 0 & \text{otherwise.} \end{cases} \quad (18)$$

Note that, by the definition of δ_U , we can also describe $E_{\hat{y}}$ as $E_{\hat{y}} = \{((i, a), (i, b)) \mid i \in [r], a, b \in D_i, (i, a) \notin \text{supp}^+(\hat{y}), (i, b) \in \text{supp}^+(\hat{y})\}$.

Algorithm for Z-free function minimization.

Step 1: Find an M^{\natural} -convex completion $(\tilde{h}_{(i, a), (j, b)})_{(i, a), (j, b) \in U}$ of $(h_{(i, a), (j, b)})_{(i, a), (j, b) \in U}$, and define $\bar{f} : \{0, 1\}^U \rightarrow \mathbf{R}$ by

$$\bar{f}(\hat{x}) = \sum_{(i, a) \in U} c_i(a) \hat{x}_{(i, a)} + \sum_{(i, a), (j, b) \in U, (i, a) \neq (j, b)} \tilde{h}_{(i, a), (j, b)} \hat{x}_{(i, a)} \hat{x}_{(j, b)} \quad (\hat{x} \in \{0, 1\}^U).$$

Step 2: Let $\hat{x}^* \in \{0, 1\}^U$ be the zero vector. While $\sum_{(i, a) \in U} \hat{x}_{(i, a)}^* < r$, do the following:

Step 2-1: Obtain $(i, a)^* \in \arg \min \{\bar{f}(\hat{x}^* + \chi_{(i, a)}) \mid (i, a) \in U \setminus \text{supp}^+(\hat{x}^*)\}$.

Step 2-2: $\hat{x}^* \leftarrow \hat{x}^* + \chi_{(i, a)^*}$.

Step 3: Let $p : V \rightarrow \mathbf{R}$ be a potential defined by $p(v) := 0$ for $v \in \{s, t\} \cup U$. Take any $\hat{y}^* \in \text{dom } \delta_U$. While $\hat{x}^* \neq \hat{y}^*$, do the following:

Step 3-1: Make the auxiliary graph $G_{\hat{x}^*, \hat{y}^*}$. Define the modified arc length $\ell_p : E_{\hat{x}^*, \hat{y}^*} \rightarrow \mathbf{R}$ by $\ell_p(u, v) := \ell(u, v) + p(u) - p(v)$ for $(u, v) \in E_{\hat{x}^*, \hat{y}^*}$.

Step 3-2: For each $v \in V$, compute the length $\Delta p(v)$ of an s - v shortest path in $G_{\hat{x}^*, \hat{y}^*}$ with respect to the modified arc length ℓ_p . Let P be an s - t shortest path having the smallest number of arcs in $G_{\hat{x}^*, \hat{y}^*}$ with respect to the modified arc length ℓ_p .

Step 3-3: For $(i, a) \in U$,

$$\hat{x}_{(i, a)}^* \leftarrow \begin{cases} \hat{x}_{(i, a)}^* - 1 & \text{if } ((i, a), (j, b)) \in P \cap E_{\hat{x}^*}, \\ \hat{x}_{(i, a)}^* + 1 & \text{if } ((j, b), (i, a)) \in P \cap E_{\hat{x}^*}, \\ \hat{x}_{(i, a)}^* & \text{otherwise,} \end{cases}$$

$$\hat{y}_{(i, a)}^* \leftarrow \begin{cases} \hat{y}_{(i, a)}^* + 1 & \text{if } ((i, a), (j, b)) \in P \cap E_{\hat{y}^*}, \\ \hat{y}_{(i, a)}^* - 1 & \text{if } ((j, b), (i, a)) \in P \cap E_{\hat{y}^*}, \\ \hat{y}_{(i, a)}^* & \text{otherwise.} \end{cases}$$

For $v \in V$, $p(v) \leftarrow p(v) + \Delta p(v)$. □

The time complexity of this algorithm is as follows, where $n := |U| = \sum_{i \in [r]} d_i$ (the proof is in Section 5).

Theorem 6. *The proposed algorithm runs in $O(nr^3 + nr \log n + n^2)$ time.*

By improving the algorithm of running time $O(n^3)$ given in [1], Cooper–Živný [2] gave an $O(n^2 \log n \log r)$ -time algorithm for minimizing Z-free functions of the form (1). Our proposed algorithm is faster than Cooper–Živný’s for some r (e.g., $r = O(n^{1/3})$).

Remark 7. At the end of Step 2, we obtain a minimizer of \bar{f}' . The validity of Step 2 is given in [10] in Japanese. For reader’s convenience we reproduce the proof of the validity here. For $\rho = 0, 1, \dots, r$ define \bar{f}'_ρ by

$$\bar{f}'_\rho(\hat{x}) := \begin{cases} \bar{f}(\hat{x}) & \text{if } \sum_{(i,a) \in U} \hat{x}_{(i,a)} = \rho, \\ +\infty & \text{otherwise.} \end{cases}$$

Then we have $\bar{f}' = \bar{f}'_r$. It suffices to show that if $\hat{x}^* \in \arg \min \bar{f}'_\rho$, then there exists $(i, a) \in U \setminus \text{supp}^+(\hat{x}^*)$ such that $\hat{x}^* + \chi_{(i,a)} \in \arg \min \bar{f}'_{\rho+1}$.

Let \hat{x} be the vector in $\arg \min \bar{f}'_{\rho+1}$ with $\|\hat{x} - \hat{x}^*\|_1$ minimum. Since $\sum_{(i,a) \in U} \hat{x}_{(i,a)} > \sum_{(i,a) \in U} \hat{x}^*_{(i,a)}$, $\text{supp}^+(\hat{x} - \hat{x}^*)$ is nonempty. Take $(i, a) \in \text{supp}^+(\hat{x} - \hat{x}^*)$. By the definition of M^1 -convexity (4), there exists $v \in \text{supp}^+(\hat{x}^* - \hat{x}) \cup \{0\}$ such that

$$\bar{f}(\hat{x}) + \bar{f}(\hat{x}^*) \geq \bar{f}(\hat{x} - \chi_{(i,a)} + \chi_v) + \bar{f}(\hat{x}^* + \chi_{(i,a)} - \chi_v). \quad (19)$$

If $v \in \text{supp}^+(\hat{x}^* - \hat{x})$, then $\hat{x} - \chi_{(i,a)} + \chi_v \in \text{dom}(\bar{f}'_{\rho+1})$ and $\hat{x}^* + \chi_{(i,a)} - \chi_v \in \text{dom}(\bar{f}'_\rho)$, which imply

$$\bar{f}(\hat{x} - \chi_{(i,a)} + \chi_v) \geq \bar{f}(\hat{x}), \quad \bar{f}(\hat{x}^* + \chi_{(i,a)} - \chi_v) \geq \bar{f}(\hat{x}^*). \quad (20)$$

By (19) and (20), we obtain $\hat{x} - \chi_{(i,a)} + \chi_v \in \arg \min \bar{f}'_{\rho+1}$. This is a contradiction to the choice of \hat{x} , since $\|(\hat{x} - \chi_{(i,a)} + \chi_v) - \hat{x}^*\|_1 = \|\hat{x} - \hat{x}^*\|_1 - 2$. Hence we have $v = 0$. Then $\bar{f}(\hat{x}^* + \chi_{(i,a)}) = \bar{f}(\hat{x})$ and therefore, $\hat{x}^* + \chi_{(i,a)} \in \arg \min \bar{f}'_{\rho+1}$.

5 Proofs

In this section, we give the proofs of Lemma 2, Theorem 3, Theorem 5, and Theorem 6.

Proof of Lemma 2. (only-if part). Suppose that there exist distinct $i, j, k \in [n]$ such that $h_{ij} < \min\{h_{jk}, h_{ik}\}$. Note that $h_{ij} < +\infty$ holds. Then

$$\begin{aligned} f(\chi_i + \chi_j) + f(\chi_k) &= h_i + h_j + h_k + h_{ij} < h_i + h_j + h_k + h_{jk} = f(\chi_j + \chi_k) + f(\chi_i), \\ f(\chi_i + \chi_j) + f(\chi_k) &= h_i + h_j + h_k + h_{ij} < h_i + h_j + h_k + h_{ik} = f(\chi_i + \chi_k) + f(\chi_j) \end{aligned}$$

hold since $h_i, h_j, h_k, h_{ij} < +\infty$. By Condition 1 of Theorem 1, f is not M^1 -convex.

Suppose that there exist distinct $i, j \in [n]$ such that $h_{ij} < 0 (< +\infty)$. Then

$$f(\chi_i + \chi_j) + f(\chi_0) = h_i + h_j + h_{ij} < h_i + h_j = f(\chi_i) + f(\chi_j)$$

holds since $h_i, h_j < +\infty$. By Condition 2 of Theorem 1, f is not M^1 -convex.

(if part). Take arbitrary distinct $i, j, k \in [n]$ and $z \in \{0, 1\}^n$ with $\text{supp}^+(z) \subseteq [n] \setminus \{i, j, k\}$. If $f(z + \chi_i + \chi_j) = +\infty$ or $f(z + \chi_k) = +\infty$ holds, then Condition 1 of Theorem 1 obviously holds. We assume $f(z + \chi_i + \chi_j) < +\infty$ and $f(z + \chi_k) < +\infty$.

It holds that

$$f(z + \chi_i + \chi_j) = f(z) + h_i + h_j + \sum_{p \in \text{supp}^+(z)} h_{ip} + \sum_{p \in \text{supp}^+(z)} h_{jp} + h_{ij}, \quad (21)$$

$$f(z + \chi_k) = f(z) + h_k + \sum_{p \in \text{supp}^+(z)} h_{kp}. \quad (22)$$

Note that all terms appearing in (21) and (22) have finite values since $f(z + \chi_i + \chi_j) < +\infty$ and $f(z + \chi_k) < +\infty$ hold. Then we have

$$\begin{aligned} & f(z + \chi_i + \chi_j) + f(z + \chi_k) \geq f(z + \chi_j + \chi_k) + f(z + \chi_i) \\ \Leftrightarrow & 2f(z) + h_i + h_j + h_k + \sum_{p \in \text{supp}^+(z)} h_{ip} + \sum_{p \in \text{supp}^+(z)} h_{jp} + \sum_{p \in \text{supp}^+(z)} h_{kp} + h_{ij} \\ & \geq 2f(z) + h_j + h_k + h_i + \sum_{p \in \text{supp}^+(z)} h_{jp} + \sum_{p \in \text{supp}^+(z)} h_{kp} + \sum_{p \in \text{supp}^+(z)} h_{ip} + h_{jk} \\ \Leftrightarrow & h_{ij} \geq h_{jk}. \end{aligned}$$

Also we have

$$\begin{aligned} & f(z + \chi_i + \chi_j) + f(z + \chi_k) \geq f(z + \chi_j + \chi_k) + f(z + \chi_i) \\ \Leftrightarrow & h_{ij} \geq h_{ik}. \end{aligned}$$

By the assumption, it holds that $h_{ij} \geq \min\{h_{jk}, h_{ik}\}$. Hence we obtain

$$f(z + \chi_i + \chi_j) + f(z + \chi_k) \geq \min\{f(z + \chi_j + \chi_k) + f(z + \chi_i), f(z + \chi_i + \chi_k) + f(z + \chi_j)\}.$$

By the assumption of $h_{ij} \geq 0$, we also obtain

$$f(z + \chi_i + \chi_j) + f(z) \geq f(z + \chi_i) + f(z + \chi_j)$$

for all distinct $i, j \in [n]$.

Proof of Theorem 3. First we give a graphical interpretation for the anti-ultrametric property. For $H := (h_{ij})_{i,j \in [n]}$ and $\alpha \in \overline{\mathbf{R}}$, let us define E_H^α and V_H^α by

$$E_H^\alpha := \{\{i, j\} \in E_H \mid h_{ij} \geq \alpha\}, \quad (23)$$

$$V_H^\alpha := \{i \mid \exists e \in E_H^\alpha \text{ such that } i \in e\}. \quad (24)$$

Let $G_H^\alpha := (V_H^\alpha, E_H^\alpha)$. Then the following lemma holds:

Lemma 8. $H := (h_{ij})_{i,j \in [n]}$ satisfies the anti-ultrametric property if and only if each connected component of G_H^α is a complete graph for every $\alpha \in \overline{\mathbf{R}}$.

Proof. (only-if part). We show the contraposition. Suppose that for some $\alpha \in \overline{\mathbf{R}}$ there exists a non-complete graph among the connected components of G_H^α . Then there exist distinct $i, j, k \in [n]$ with $\{i, j\}, \{j, k\} \in E_H^\alpha \not\subseteq \{i, k\}$. By the definition of E_H^α , it holds that $\min\{h_{ij}, h_{jk}\} \geq \alpha > h_{ik}$. This means that $\{h_{ij}, h_{jk}, h_{ik}\}$ does not satisfy the anti-ultrametric property.

(if part). Suppose that each connected component of G_H^α is a complete graph for all $\alpha \in \overline{\mathbf{R}}$. To show the anti-ultrametric property of $(h_{ij})_{i,j \in [n]}$, it suffices to prove $h_{jk} = h_{ik}$ for all distinct i, j, k satisfying $h_{ij} > h_{jk}$. If $h_{ik} \geq h_{ij}$, then there exists a non-complete graph among the connected components of G_H^α for $\alpha = h_{ij}$, which is a contradiction. If $h_{ij} > h_{ik} > h_{jk}$, then there exists a non-complete graph among the connected components of G_H^α for $\alpha = h_{ik}$, which is a contradiction. If $h_{jk} > h_{ik}$, then there exists a non-complete graph among the connected components of G_H^α for $\alpha = h_{jk}$, which is a contradiction. Therefore we must have $h_{jk} = h_{ik}$. \square

We are now ready to prove Theorem 3.

Proof of Theorem 3. (only-if part). Suppose to the contrary that $H := (h_{ij})_{i,j \in [n]}$ is \mathbf{M}^\natural -convex completable and that there exists a chordless cycle C of G_H with $|\arg \min_{e \in C} w(e)| = 1$. Let $C = \{\{i_1, i_2\}, \{i_2, i_3\}, \dots, \{i_m, i_1\}\}$, and consider the corresponding entries $\{h_{i_1 i_2}, h_{i_2 i_3}, \dots, h_{i_m i_1}\}$ of H . Note that $h_{i_p i_q}$ is undefined for $p, q \in [m]$ with $|p - q| \neq 1 \pmod m$. We may assume $\alpha := h_{i_1 i_2} = \min\{h_{i_1 i_2}, h_{i_2 i_3}, \dots, h_{i_m i_1}\}$. By the assumption of $|\arg \min_{e \in C} w(e)| = 1$, we have $\min\{h_{i_2 i_3}, \dots, h_{i_m i_1}\} > \alpha$. Since $\{h_{i_1 i_2}, h_{i_2 i_3}, h_{i_1 i_3}\}$ should satisfy the anti-ultrametric property, we have to assign α to $h_{i_1 i_3}$ to obtain an \mathbf{M}^\natural -convex completion. Since $\{h_{i_1 i_3}, h_{i_3 i_4}, h_{i_1 i_4}\}$ should satisfy the anti-ultrametric property, we have to assign α to $h_{i_1 i_4}$ to obtain an \mathbf{M}^\natural -convex completion. By repeating this procedure, we arrive at $h_{i_1 i_{m-1}} = \alpha$. This is a contradiction, since $h_{i_1 i_{m-1}} < \min\{h_{i_{m-1} i_m}, h_{i_1 i_m}\}$ and hence the anti-ultrametric property fails for $\{h_{i_1 i_{m-1}}, h_{i_{m-1} i_m}, h_{i_1 i_m}\}$.

(if part). For $\alpha \in \overline{\mathbf{R}}_+$, define S_H^α by

$$S_H^\alpha := \{\{i, j\} \notin E_H \mid i, j \in V_H^\alpha, i \text{ and } j \text{ are connected in } G_H[E_H^\alpha]\},$$

where $G_H[E_H^\alpha]$ is the subgraph of G_H induced by E_H^α . Let $\tilde{G}_H^\alpha := (V_H^\alpha, E_H^\alpha \cup S_H^\alpha)$. Recall that E_H^α and V_H^α are defined in (23) and (24).

First we show that if each connected component of \tilde{G}_H^α is a complete graph for every $\alpha \in \overline{\mathbf{R}}_+$, then H is \mathbf{M}^\natural -convex completable. Let $\alpha_1 > \alpha_2 > \dots > \alpha_m$ be the distinct values of defined elements of $(h_{ij})_{i,j \in [n]}$ (α_1 can be the infinite value). Note that for each undefined element h_{ij} , $\{i, j\}$ is in S_α for some $\alpha \in \{\alpha_1, \alpha_2, \dots, \alpha_m\}$. We assign α_1 to each undefined element h_{ij} such that $\{i, j\} \in S_H^{\alpha_1}$, and α_k to each h_{ij} such that $\{i, j\} \in S_H^{\alpha_k} \setminus S_H^{\alpha_{k-1}}$. Then we obtain a certain completion $\tilde{H} := (\tilde{h}_{ij})_{i,j \in [n]}$ of $(h_{ij})_{i,j \in [n]}$. It is clear that each connected component of G_H^α is a complete graph for every $\alpha \in \overline{\mathbf{R}}$. By Lemma 8, \tilde{H} satisfies the anti-ultrametric property. This means that H is \mathbf{M}^\natural -convex completable.

Next we show that if $|\arg \min_{e \in C} w(e)| \geq 2$ holds for every chordless cycle C of G_H , then each connected component of \tilde{G}_H^α is a complete graph for every $\alpha \in \overline{\mathbf{R}}_+$. Take arbitrary $\alpha \in \overline{\mathbf{R}}_+$ and i and j which are connected in \tilde{G}_H^α (Note that vertex sets of connected components of \tilde{G}_H^α are the same as those of $G_H[E_H^\alpha]$). It suffices to prove that $\{i, j\} \in E_H^\alpha$ or $\{i, j\} \in S_H^\alpha$ holds. Suppose to the contrary that there exist i and j such that $\{i, j\} \notin E_H^\alpha$ and $\{i, j\} \notin S_H^\alpha$ hold. Let I be the set of such $\{i, j\}$. Let $\{i_0, j_0\} \in I$ be a pair of vertices such that the number of edges of a shortest i_0 - j_0 path on $G_H[E_H^\alpha]$ is minimum in I . Since i_0 and j_0 are connected in $G_H[E_H^\alpha]$ and $\{i_0, j_0\} \notin S_H^\alpha$, we have $\{i_0, j_0\} \in E_H$. Moreover since $\{i_0, j_0\} \notin E_H^\alpha$, $h_{i_0 j_0} < \alpha$ holds. Take a i_0 - j_0 shortest path P_0 . Then $P_0 \cup \{i_0, j_0\}$ is a chordless cycle of G_H . Indeed, if $P_0 \cup \{i_0, j_0\}$ has a chord in G_H , there exist i' and j' satisfying $\{i', j'\} \neq \{i_0, j_0\}$ in $P_0 \cup \{i_0, j_0\}$ such that $E_H^\alpha \not\ni \{i', j'\} \in E_H$ by the minimality of $|I|$. Then $\{i', j'\} \in I$ and the number of edges of a shortest i' - j' path is smaller than those of P_0 . However this is a contradiction to the minimality of $\{i_0, j_0\}$. Hence $P_0 \cup \{i_0, j_0\}$ is a chordless cycle of G_H . It holds that $h_{ij} \geq \alpha$ for $\{i, j\} \in P_0$ and $h_{i_0 j_0} < \alpha$. Therefore we obtain $|\arg \min_{e \in P_0 \cup \{i_0, j_0\}} w(e)| = 1$. This contradicts the assumption of $|\arg \min_{e \in C} w(e)| \geq 2$. Hence we have $\{i, j\} \in E_H^\alpha$ or $\{i, j\} \in S_H^\alpha$. \square

Proof of Theorem 5. Let $H := (h_{(i,a),(j,b)})_{(i,a),(j,b) \in U}$, where the entries $h_{(i,a),(j,b)}$ with $i = j$ are undefined. Recall that $G_H = (U, E_H; w)$ is the assignment graph of H . By the definition of E_H and $h_{(i,a),(j,b)}$ in (11), we have $E_H = \{(i,a), (j,b) \mid i \neq j, a \in D_i, b \in D_j\}$. By Theorem 3, H is M^\natural -convex completable if and only if every chordless cycle C satisfies the condition $|\arg \min_{e \in C} w(e)| \geq 2$.

First we show that chordless cycles in G_H have length 3 or 4. Take any chordless cycle $C = \{(i_1, a_1), (i_2, a_2)\}, \{(i_2, a_2), (i_3, a_3)\}, \dots, \{(i_k, a_k), (i_1, a_1)\}$ of G_H . Since C is chordless, we have $i_1 = i_p$ for $3 \leq p \leq k-1$ and $i_2 = i_q$ for $4 \leq q \leq k$. This implies $k \leq 4$, since otherwise we obtain $i_1 = i_4 = i_2$, contradicting the existence of an edge between (i_1, a_1) and (i_2, a_2) .

For a (chordless) cycle of length 3, say, $C = \{(i_1, a_1), (i_2, a_2)\}, \{(i_2, a_2), (i_3, a_3)\}, \{(i_3, a_3), (i_1, a_1)\}$ with $i_1 \neq i_2 \neq i_3 \neq i_1$, the condition $|\arg \min_{e \in C} w(e)| \geq 2$ is equivalent to (2) for JWP. For a chordless cycle of length 4, say, $C = \{(i_1, a_1), (i_2, a_2)\}, \{(i_2, a_2), (i_3, a_3)\}, \{(i_3, a_3), (i_4, a_4)\}, \{(i_4, a_4), (i_1, a_1)\}$ we have $i_1 \neq i_2, i_3 \neq i_4, i_1 = i_3, i_2 = i_4, a_1 \neq a_3, a_2 \neq a_4$, and then the condition $|\arg \min_{e \in C} w(e)| \geq 2$ is equivalent to (3) for Z-freeness.

Proof of Theorem 6. We investigate each step in turn.

(Step 1). Since the number of defined elements of $(h_{(i,a),(j,b)})_{(i,a),(j,b) \in U}$ is $O(n^2 - \sum_{i=1}^r d_i^2) = O(n^2)$, we can find an M^\natural -convex completion in $O(n^2 + n \log n)$ time (recall Remark 4).

(Step 2). If we have the value of $\bar{f}(\hat{x}^*)$, we can compute the value of $\bar{f}(\hat{x}^* + \chi_{(i,a)})$ in $O(r)$ time since $\bar{f}(\hat{x}^* + \chi_{(i,a)}) = \bar{f}(\hat{x}^*) + c_i(a) + \sum_{(j,b) \in \text{supp}^+(\hat{x}^*)} \tilde{h}_{(i,a),(j,b)}$. Hence the time complexity of Step 3 is $O(nr^2)$ time.

(Step 3). Recall the definition of $G_{\hat{x}^*, \hat{y}^*}$ in (12)–(18). We have $|E_{\hat{x}^*}| = O(r(n-r)) = O(nr)$, $|E_{\hat{y}^*}| = O(n)$, $|E^+| = O(r)$, and $|E^-| = O(r)$. Hence $|E_{\hat{x}^*, \hat{y}^*}| = O(nr)$. Furthermore we need to compute ℓ only on $E_{\hat{x}^*}$, since ℓ is equal to zero on other arcs. If we have the value of $\bar{f}(\hat{x}^*)$ at hand, we can compute the value of $\bar{f}(\hat{x}^* + \chi_{(j,b)} - \chi_{(i,a)})$ in $O(r)$ time since

$$\begin{aligned} & \bar{f}(\hat{x}^* + \chi_{(j,b)} - \chi_{(i,a)}) \\ &= \bar{f}(\hat{x}^*) - \left(c_i(a) + \sum_{(k,c) \in \text{supp}^+(\hat{x}^*)} \tilde{h}_{(i,a),(k,c)} \right) + \left(c_j(b) + \sum_{(k,c) \in \text{supp}^+(\hat{x}^* - \chi_{(i,a)})} \tilde{h}_{(j,b),(k,c)} \right). \end{aligned}$$

Therefore we can construct the auxiliary graph $G_{\hat{x}^*, \hat{y}^*}$ in $O(nr^2)$ time.

The modified arc length ℓ_p is nonnegative [6, Section 5.2]. Hence we can compute $\Delta p(v)$ for $v \in V$ and a shortest path P in Step 3-2 in $O(nr + n \log n)$ time by using Dijkstra's algorithm with Fibonacci heaps [4] (see also [12, Section 7.4]). We can update \hat{x}^* , \hat{y}^* , and p in Step 3-3 in $O(nr)$ time. By one iteration of Step 3, the value of $\|\hat{x}^* - \hat{y}^*\|_1$ is decreased by two. Hence the number of iterations of Step 3 is bounded by $O(r)$. Therefore the time complexity of Step 3 is $O(nr^3 + nr \log n)$.

By the above discussion, we see that the proposed algorithm runs in $O(nr^3 + nr \log n + n^2)$ time.

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